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PLASTIC DEFORMATION AT THE TIP OF AN EDGE CRACK

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An exact analytic solution of the problem of a crack emerging at the boundary of a free half-plane, is given for the case when the plastic zone is concentrated at the extension of the crack (a plastic zone appears in precisely such a form in such materials as low- C steel [1 —3]. In particular, when the plastic zone has zero length, we arrive at the problem of stretching of an elastic half-plane with an edge crack. A general formula is obtained for the stress intensity coefficient K_I, and the formula yields the results of [4 — 6] as particular cases. An approximate solution of the problem discussed below was constructed in [7] for a particular case of a constant normal stress σ_y at infinity.

1. Formulation of the problem. We consider a plate with an edge crack of length l. We assume that the plate material is perfect, elastoplastic, and satisfies the Tresca — St. Venant condition of plasticity and that the deformations are small. We represent the crack in the form of a mathematical cut of zero thickness. For this reason a plastic region will form at the tip of the crack when arbitrarily small external loads are applied, and the size of this region will increase with increasing loads. We assume that the plastic deformation is concentrated along a narrow rectilinear slippage plane along the continuation of the edge crack.

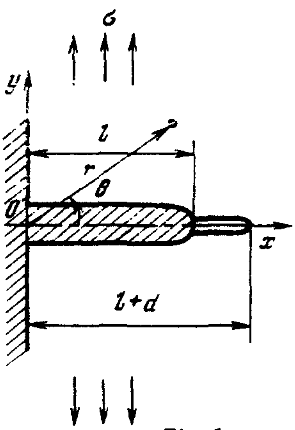

Fig. 1

Let us construct a solution of the following boundary value problem:

$$\theta = \pm \pi / 2, \quad \sigma_\theta = \tau_{r\theta} = 0 \tag{1.1}$$

$$\theta = 0, \quad 0 < r < l, \quad \tau_{r\theta} = 0 \tag{1.2}$$

$$\sigma_\theta = -\sigma(x)$$
$$\theta = 0, \quad l < r < l + d, \quad \tau_{r\theta} = 0, \quad \sigma_\theta = \sigma_s - \sigma(x)$$
$$\theta = 0, \quad r > l + d, \quad \tau_{r\theta} = 0, \quad \partial u_\theta / \partial r = 0$$
$$\theta = 0, \quad [\sigma_\theta] = [\tau_{r\theta}] = 0 \tag{1.3}$$

$$r \to \infty, \quad \sigma_\theta \to 0, \quad \sigma_r \to 0, \quad \tau_{r\theta} \to 0 \tag{1.4}$$

Here σ_θ, σ_r and $\tau_{r\theta}$ are the components of the stress tensor in polar r, θ -coordinates; u_θ and u_r are the components of the displacement vector; square brackets denote a jump in the value of the quantity appearing within these brackets; σ_s is the yield point and $\sigma(x)$ is a given function. In particular, when $\sigma(x) = \sigma = \text{const}$, we have the problem depicted in Fig. 1.

2. Derivation of the Wiener–Hopf equation. Applying the integral Mellin transform

$$f^*(p) = \int_0^\infty f(r)\, r^p dr$$

(where p is a complex parameter) to the equation of equilibrium and the condition of continuity, we obtain [8]

$$\frac{d^4\sigma_\theta^*}{d\theta^4} + [(p+1)^2 + (p-1)^2]\frac{d^2\sigma_\theta^*}{d\theta^2} + (p+1)^2(p-1)^2\sigma_\theta^* = 0 \qquad (2.1)$$

The functions σ_r^* and $\tau_{r\theta}^*$ can be written in terms of σ_θ^* as follows:

$$\tau_{r\theta}^* = \frac{1}{p-1}\frac{d\sigma_\theta^*}{d\theta}, \quad p\sigma_r^* = \frac{1}{p-1}\frac{d^2\sigma_\theta^*}{d\theta^2} - \sigma_\theta^* \qquad (2.2)$$

The solution of (2.1) has the form (here and henceforth the upper sign is used when $0 \leqslant \theta \leqslant \pi/2$, and the lower sign when $-\pi/2 \leqslant \theta \leqslant 0$)

$$\sigma_\theta^*(p,\theta) = C^\pm(p)\left[\cos(p+1)\left(\frac{\pi}{2}\mp\theta\right) - \cos(p-1)\left(\frac{\pi}{2}\mp\theta\right)\right] + D^\pm(p)\left[\sin(p+1)\left(\frac{\pi}{2}\mp\theta\right) - \frac{p+1}{p-1}\sin(p-1)\left(\frac{\pi}{2}\mp\theta\right)\right] \qquad (2.3)$$

The unknown functions $C^\pm(p)$ and $D^\pm(p)$ are determined from the remaining boundary conditions.

In accordance with the formulas (2.2), (2.3) and the Hooke's Law, we obtain

$$\sigma_\theta^*(p,\theta) = C^\pm(p)\left[\cos(p+1)\left(\frac{\pi}{2}\mp\theta\right) - \cos(p-1)\left(\frac{\pi}{2}\mp\theta\right)\right] + D^\pm(p)\left[\sin(p+1)\left(\frac{\pi}{2}\pm\theta\right) - \frac{p+1}{p-1}\sin(p-1)\left(\frac{\pi}{2}\mp\theta\right)\right] \qquad (2.4)$$

$$\tau_{r\theta}^*(p,\theta) = \frac{1}{p-1}\Big\{C^\pm(p)\Big[(p\pm1)\sin(p\pm1)\left(\frac{\pi}{2}\mp\theta\right) - (p\mp1)\sin(p\mp1)\left(\frac{\pi}{2}\mp\theta\right)\Big] \mp D^\pm(p)(p+1)\times \Big[\cos(p+1)\left(\frac{\pi}{2}\mp\theta\right) - \cos(p-1)\left(\frac{\pi}{2}\mp\theta\right)\Big]\Big\}$$

$$\left(\frac{\partial u_\theta}{\partial r}\right)^* = \frac{1+\nu}{(p-1)E}\Big\{\pm4(1-\nu)\Big[-C^\pm(p)\sin(p+1)\left(\frac{\pi}{2}\mp\theta\right) + D^\pm(p)\cos(p+1)\left(\frac{\pi}{2}\mp\theta\right)\Big] + (p-1)\,\tau_{r\theta}^*(p,\theta)\Big\}$$

When $\theta = 0$ we have $[\sigma_\theta^*] = [\tau_{r\theta}^*] = 0$, $\tau_{r\theta}^* = 0$, and this implies

$$C^+ = C^-, \quad D^+ = D^-, \quad D^+ = -\frac{pC^+}{p+1}\operatorname{ctg}\frac{p\pi}{2} \qquad (2.5)$$

From (2.4) and (2.5) we obtain

$$\sigma_\theta^*(p,0) = -\frac{2C^+}{p^2-1}\frac{p^2-\sin^2 p\pi/2}{\sin p\pi/2} \qquad (2.6)$$

$$\left[\left(\frac{\partial u_\theta}{\partial r}\right)^*\right]\Big|_{\theta=0} = -\frac{4(1-\nu^2)}{E}\frac{2C^+}{p^2-1}\cos p\frac{\pi}{2}$$

Eliminating from (2. 6) the function $C^+(p)$ and taking (1. 2) into account, we arrive at the inhomogeneous Wiener — Hopf equation

$$\Phi^-(p) = K(p)\,G(p)\,(l+d)\,[Q(p) + \Phi^+(p)] \qquad (2.7)$$

where

$$\Phi^-(p) = -\frac{E}{4(1-\nu^2)}\int_0^1 \left[\frac{\partial u_\theta(lt+dt,0)}{\partial t}\right]\Big|_{\theta=0} t^p dt$$

$$K(p) = \operatorname{ctg} p\frac{\pi}{2}$$

$$\Phi^+(p) = \int_1^\infty \sigma_\theta(lt+dt,0)\,t^p dt, \quad G(p) = \frac{\sin^2 p\pi/2}{\sin^2 p\pi/2 - p^2}$$

$$Q(p) = \left(1+\frac{d}{l}\right)^{-(p+1)}\left\{\frac{\sigma_s}{p+1}\left[\left(1+\frac{d}{l}\right)^{p+1} - 1\right] - \int_0^{1+d/l} \sigma(lt)\,t^p dt\right\}$$

3. **Solution of the boundary value problem.** The functional equation (2. 7) exists in the strip $-1 < \operatorname{Re} p < 0$, $-\infty < \operatorname{Im} p < \infty$. The function $G(p)$ can be written in the strip $-1 < \operatorname{Re} p < 0$, $-\infty < \operatorname{Im} p < \infty$ in the form [9]

$$G(p) = \frac{G^+(p)}{G^-(p)} \qquad (3.1)$$

$$G^\pm(p) = \exp\left[\frac{1}{2\pi i}\int_{a^\pm - i\infty}^{a^\pm + i\infty} \frac{\ln G(t)}{t-p}\,dt\right] \quad (-1 < a^- < \operatorname{Re} p < a^+ < 0)$$

Here $G^+(p)$ and $G^-(p)$ are entire function, analytic and without zeros in the regions $\operatorname{Re} p < 0$ and $\operatorname{Re} p > -1$ respectively. Both functions tend to unity at infinity. We shall write the function $K(p)$ in the form [10]

$$K(p) = 2p^{-1}K^+(p)K^-(p), \quad K^\pm(p) = \Gamma(1 \mp p/2)\,/\,\Gamma(1/2 \mp p/2) \qquad (3.2)$$

Factorizing (3. 2) and (3. 1), we can write the functional equation (2. 7) as follows:

$$\frac{G^-(p)}{K^-(p)}\Phi^-(p) = \frac{2(l+d)}{p}K^+(p)\,G^+(p)[\Phi^+(p) + Q(p)] \qquad (3.3)$$

Consider the function

$$\varphi(p) = p^{-1}K^+(p)\,G^+(p)\,Q(p)$$

Let the function $\varphi(p)$ possess the following properties [10]:

(a) be analytic and regular in the strip $-1 < a^- < \operatorname{Re} p < a^+ < 0$, $-\infty < \operatorname{Im} p < \infty$, and

(b) $|\varphi(p)| < A\,|\operatorname{Im} p|^{-\alpha}$ $(\alpha > 0)$ as $|\operatorname{Im} p| \to \infty$, with the inequality holding uniformly for all $\operatorname{Re} p$ in the strip $a^- + \varepsilon < \operatorname{Re} p < a^+ - \varepsilon$, $\varepsilon > 0$.

Then we can write the function $\varphi(p)$ in the strip $-1 < a^- < a_1^- < \operatorname{Re} p < a_1^+ < a^+ < 0$, $|\operatorname{Im} p| < \infty$ in the form

$$\varphi(p) = \varphi^+(p) - \varphi^-(p), \quad \varphi^\pm(p) = \frac{1}{2\pi i} \int_{a_1^\pm - i\infty}^{a_1^\pm + i\infty} \frac{\varphi(t)}{t-p} dt \tag{3.4}$$

The functions $\varphi^+(p)$ and $\varphi^-(p)$ are regular and have no zeros in the regions $\operatorname{Re} p < a_1^+$ and $\operatorname{Re} p > a_1^-$ respectively. Substituting (3.4) into (3.3), we obtain

$$\frac{G^-(p)}{K^-(p)} \Phi^-(p) + 2(l+d)\varphi^-(p) = \frac{2(l+d)}{p} K^+(p) G^+(p) \Phi^+(p) + 2(l+d)\varphi^+(p) \tag{3.5}$$

This yields, in accordance with the properties of the functions $G^\pm(p)$, $K^\pm(p)$, $\Phi^\pm(p)$ and $\varphi^\pm(p)$,

$$\Phi^+(p) = -\frac{p\varphi^+(p)}{K^+(p)G^+(p)}, \quad \Phi^-(p) = -2(l+d)\frac{\varphi^-(p)K^-(p)}{G^-(p)} \tag{3.6}$$

Using now (2.6) we determine the function $C^+(p)$, find the Mellin transform of the stresses in question, and inverting the transform, the stresses themselves. Next we shall consider some particular cases of the general solution (3.6).

Case of constant load. Let the function $\sigma(x)$ be constant: $\sigma(x) = \sigma = \text{const}$ (see Fig. 1). We also have

$$Q(p) = \frac{1}{p+1}\left[\sigma_s - \sigma - \sigma_s \left(\frac{l}{l+d}\right)^{p+1}\right]$$

We can write the functions $\varphi^+(p)$ and $\varphi^-(p)$ in accordance with the properties of the Cauchy-type integrals, in the form

$$\varphi^+(p) = \frac{\sigma_s - \sigma}{p(p+1)} K^+(p) G^+(p) \left[1 + \frac{p\sqrt{\pi}G^+(-1)}{2K^+(p)G^+(p)}\right] - \sigma_s \gamma^+(p) \tag{3.7}$$

$$\varphi^-(p) = (\sigma_s - \sigma)\frac{\sqrt{\pi}G^+(-1)}{2(p+1)} - \sigma_s\gamma^-(p)$$

where

$$\gamma^\pm(p) = \frac{1}{2\pi i}\int_{a_1^\pm - i\infty}^{a_1^\pm + i\infty} \frac{K^+(t)G^+(t)}{t(t+1)(1+d/l)^{t+1}} \frac{dt}{t-p} \tag{3.8}$$

To find the quantity d, we consider the function

$$\Phi^+(p) = \int_1^\infty \sigma_\theta[(l+d)t, 0] t^p dt \tag{3.9}$$

From this, using the condition

$$\sigma_\theta[(l+d)t, 0] = \frac{K_I}{\sqrt{2\pi(l+d)(t-1)}} \quad (t \to +1+0)$$

we find, according to an Abel-type theorem,

$$\Phi^+(p) = \frac{K_I}{\sqrt{-2(l+d)p}} \quad (p \to \infty) \tag{3.10}$$

On the other hand, when $p \to \infty$, we have from (3.6) and (3.7)

$$\Phi^+(p) = \left[\sqrt{\frac{\pi}{2}}(\sigma - \sigma_s)G^+(-1) - \sqrt{2}\,\sigma_s g\left(\frac{d}{l}\right)\right]\frac{1}{\sqrt{-p}} \tag{3.11}$$

$$g\left(\frac{d}{l}\right) = \frac{1}{2\pi i}\int_{a_1^+ - i\infty}^{a_1^+ + i\infty} \frac{K^+(t)\,G^+(t)}{t(t+1)}\,\frac{dt}{(1 + d/l)^{t+1}}$$

The conditions of boundedness of the stresses at the tip of the plastic line yield the relation

$$\frac{\sqrt{\pi}}{2}G^+(-1)\left(\frac{\sigma}{\sigma_s} - 1\right) = g\left(\frac{d}{l}\right)$$

and Fig. 2 depicts the dependence of d/l on σ/σ_s

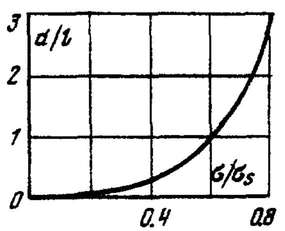

Fig. 2

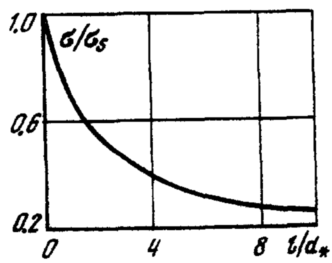

Fig. 3

Opening of the crack is of interest from the point of view of the fracture mechanics. This is determined from the formula

$$v_0 = \int_{d+l}^{x}\left[\frac{\partial u_\theta}{\partial r}\right]\Bigg|_{\theta=0} dr \tag{3.12}$$

where

$$\left[\frac{\partial u_\theta}{\partial r}\right]\Bigg|_{\theta=0} = \frac{1}{2\pi i}\int_L \left[\left(\frac{\partial u_\theta}{\partial r}\right)^*\right]\Bigg|_{\theta=0} r^{-p-1}\,dp \tag{3.13}$$

$$(L\colon\ a_1^- < \operatorname{Re} p = c < a_1^+,\ -\infty < \operatorname{Im} p < \infty)$$

In accordance with the second formulas of (3.6), (3.7) and (3.13) we find

$$\left[\frac{\partial u_\theta}{\partial r}\right]\Bigg|_{\theta=0} = -\frac{8(1-\nu^2)}{E}\sigma_s(d+l)\frac{1}{2\pi i}\int_L \frac{(l+d)^p K^-(p)}{(p+1)G^-(p)} \times \tag{3.14}$$

$$\left[\left(\frac{\sigma}{\sigma_s} - 1\right)\frac{\sqrt{\pi}}{2}G^+(-1) + (p+1)\gamma^-(p)\right] r^{-p-1}\,dp$$

Substituting (3.14) into (3.12), we obtain

$$v_0 = \frac{8(1-\nu^2)}{E}\sigma_s(l+d)\left\{\frac{\sigma}{2\sigma_s}\frac{G^+(-1)}{G^-(0)} + \frac{2\gamma^-(0) - \sqrt{\pi}G^+(-1)}{2\sqrt{\pi}G^-(0)} + \right.$$
$$\left.\frac{1}{2\pi i}\int_L\left(\frac{l}{x}\right)^p \frac{(1+d/l)^p K^-(p)}{p(p+1)G^-(p)}\left[(p+1)\gamma^-(p) + g\left(\frac{d}{l}\right)\right]dp\right\} \quad (x \neq 0)$$

At the tip of the crack (where $x = l$) its opening is given by

$$\frac{d_*}{l} = \pi\left(1+\frac{d}{l}\right)\left[\frac{\sigma}{\sigma_s}\frac{G^+(-1)}{2G^-(0)} + \left(\frac{\sigma}{\sigma_s}-1\right)\frac{\sqrt{\pi}}{2}G^+(-1)\times\right.$$
$$\frac{1}{2\pi i}\int_L \frac{(1+d/l)^p K^-(p)}{p(p+1)G^-(p)}dp + \frac{1}{2\pi i}\int_L \frac{(1+d/l)^p K^-(p)}{pG^-(p)}\gamma^-(p)\,dp +$$
$$\left.\frac{2\gamma^-(0) - \sqrt{\pi}G^+(-1)}{2\sqrt{\pi}G^-(0)}\right] \quad \left(d_* = \frac{\pi E v_0}{8(1-\nu^2)\sigma_s}\right)$$

Figure 3 shows the dependence of the function σ/σ_s on l/d_*. Let $d = 0$. In this case

$$g(0) = -{}^1\!/_2\sqrt{\pi}G^+(-1) \tag{3.15}$$

We note that the function $G^+(-1)$ can be calculated with any prescribed degree of accuracy. For example, computing the value of $G^+(-1)$ to the sixth decimal place yields $G^+(-1) = 1.121524$.

Substituting the expression (3.15) into (3.11) and equating two asymptotics of the function $\Phi^+(p)$ when $d = 0$, we obtain the stress intensity coefficient

$$K_I = \sigma\sqrt{\pi l}\,G^+(-1)$$

The above result coincides with the known expression (see e.g. [4, 6]). In particular, we can also determine the opening of the crack

$$v_0 = \frac{4(1-\nu^2)}{E}\sigma l\sqrt{\pi}G^+(-1)\left[\frac{1}{\sqrt{\pi}G^-(0)} + \right.$$
$$\left.\frac{1}{2\pi i}\int_L\left(\frac{l}{x}\right)^p\frac{K^-(p)}{p(p+1)G^-(p)}dp\right] \quad (x\neq 0)$$

C a s e o f a l i n e a r l o a d. Let the function $\sigma(x)$ have the form

$$\sigma(x) = \sigma_1 + \sigma_2\frac{c_* - x}{c_*}$$

Here σ_1, σ_2 and c_* are given constants. In this case we have

$$Q(p) = \frac{1}{p+1}\left[(\sigma_s - \sigma_1 - \sigma_2) - \sigma_s\left(\frac{l}{l+d}\right)^{p+1}\right] + \frac{\sigma_2(l+d)}{c_*(p+2)}$$

The functions $\varphi^+(p)$ and $\varphi^-(p)$ now become, in accordance with the theory of the Cauchy-type integrals,

$$\varphi^+(p) = \frac{\sigma_s - \sigma_1 - \sigma_2}{p(p+1)}K^+(p)G^+(p)\left[1 + \frac{p\sqrt{\pi}G^+(-1)}{2K^+(p)G^+(p)}\right] + \tag{3.16}$$
$$\frac{\sigma_2(l+d)K^+(p)G^+(p)}{p(p+2)c_*}\left[1 + \frac{pG^+(-2)}{\sqrt{\pi}K^+(p)G^+(p)}\right] - \sigma_s\gamma^+(p)$$
$$\varphi^-(p) = \frac{(\sigma_s - \sigma_1 - \sigma_2)\sqrt{\pi}}{2(p+1)}G^+(-1) + \frac{\sigma_2(l+d)G^+(-2)}{\sqrt{\pi}(p+2)c_*} - \sigma_s\gamma^-(p)$$

Substituting (3.16) into (3.6), we obtain

$$\Phi^+(p) = \frac{\sigma_1 + \sigma_2 - \sigma_s}{p+1}\left[1 + \frac{p\sqrt{\pi}\,G^+(-1)}{2K^+(p)\,G^+(p)}\right] - \frac{\sigma_2(l+d)}{c_*(p+2)}\left[1 + \frac{pG^+(-2)}{\sqrt{\pi}\,K^+(p)\,G^+(p)}\right] + \frac{\sigma_s p\gamma^+(p)}{K^+(p)\,G^+(p)} \tag{3.17}$$

$$\Phi^-(p) = 2(l+d)\,\sigma_s\frac{K^-(p)}{G^-(p)}\gamma^-(p) - (l+d)\frac{(\sigma_s - \sigma_1 - \sigma_2)\sqrt{\pi}\,G^+(-1)}{(p+1)\,G^-(p)}K^-(p) - \frac{2\sigma_2(l+d)^2\,G^+(-2)}{c_*(p+2)\sqrt{\pi}\,G^-(p)}K^-(p)$$

The quantity d is determined using the method given in the previous example.

$$\left(\frac{\sigma_1+\sigma_2}{\sigma_s} - 1\right)\frac{\sqrt{\pi}}{2}G^+(-1) - \frac{\sigma_2(l+d)}{\sqrt{\pi}\,\sigma_s c_*}G^+(-2) = g\left(\frac{d}{l}\right)$$

Let $d = 0$ and $c_* = l$ (Koiter's problem [4]). Let us obtain the stress intensity coefficient K_{I}. This, in accordance with (3.17) and (3.10) has the form

$$K_{\text{I}} = (\sigma_1 + \sigma_2)\sqrt{\pi l}\,G^+(-1) - 2\sigma_2\sqrt{\frac{l}{\pi}}\,G^+(-2) \tag{3.18}$$

Substituting the numerical values obtained for the functions $G^+(-1)$ and $G^+(-2)$ on a digital computer into (3.18), we obtain

$$K_{\text{I}} = (1.1215\sigma_1 + 0.4391\,\sigma_2)\sqrt{\pi l}$$

which agrees with the result given in [4].

Case of an arbitrary, symmetrical normal load. Let $d = 0$. The stress intensity coefficient is given, according to the formulas (3.6) and (3.10), by

$$K_{\text{I}} = -\frac{\sqrt{l}}{\pi i}\int_{a_1+-i\infty}^{a_1++i\infty}\frac{K^+(t)}{t}G^+(t)\int_0^1\sigma(l\tau)\,\tau^t\,d\tau\,dt \tag{3.19}$$

Formula (3.19) yields the results of [4 — 6] et al. (see [1]) as particular cases. Let now $d \neq 0$. In this case we have the following formula for determining d:

$$\frac{\sigma_s\sqrt{\pi}}{2}G^+(-1) + \frac{1}{2\pi i}\int_{a_1+-i\infty}^{a_1++i\infty}\frac{K^+(t)\,G^+(t)}{t(1+d/l)^{t+1}}\int_0^{1+d/l}\sigma(l\tau)\,\tau^t\,d\tau\,dt = -\sigma_s g\left(\frac{d}{l}\right)$$

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